Exact Meridian Length of an Infinite Family of 4-Braid

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Abstract

A significant challenge in knot theory is the computation of geometric invariants, as the computational complexity increases exponentially with the crossing number. This paper presents a closed algebraic formula for the meridian length of an infinite family of alternating links with two components. The infinite family is obtained by the closure of the 4-braids $\left(\sigma_2^{-1}\sigma_1\sigma_3\sigma_2^{-1}\right)^n$ for odd $n \geq 3$.

Keywords: Knot theory, knot invariant, maximal cusp, braid.

MSC (2020): 57K10; 57M25.

1 Introduction

For a link in S^3 whose complement admits a complete hyperbolic structure, the Mostow-Prasad rigidity theorem shows the uniqueness of this structure. Consequently, the complete hyperbolic structure is an invariant of the link. Each component of the link has a cusp neighborhood with a torus boundary. The hyperbolic structure of the link complement induces a Euclidean structure on this boundary torus. The hyperbolic structure yields several invariants, including hyperbolic volume, cusp shapes, cusp area, and the lengths of geodesic, etc.

A central question in this context is finding a closed geodesic in the link complement that minimizes length. The shortest such geodesic is called *meridian*, which travels around the cusp boundary of link complement. By 2π-theorem [1], the length of the meridian on the torus boundary is bounded above by 2π. Agol [2] and Lackenby [3] further improved this bound to 6. Many other attempts have been made to refine this upper bound. Agol [2] constructed a family of knots by performing Dehn fillings on one component of the Borromean ring, and the meridian length of this family approaches to 4.

It is well-known that any knot can be obtained by performing Dehn filling on a fully augmented link. The length of the meridian in fully augmented links is bounded above by 2 due to the work of Futer and Purcell [4] and Schoenfield [5]. Furthermore, Purcell [6] has constructed a family of knots via Dehn fillings on generalized augmented links, and the meridian lengths of this family approach 4 from below.

Adams et al. [7] established bounds on meridian lengths of knots in terms of crossing number n. The upper bound on meridian lengths is $6 - 7/n$ for general knots and $3 - 6/n$ for alternating knots. The family of 2-bridge knots have the meridian length less than 2 [8]. Thistlethwaite and Tsvietkova have discussed an infinite family of 3-braids $\left(\sigma_1\sigma_2^{-1}\right)^n, n \geq 3$

and shown that the meridian length of this family converges to $\sqrt{3}$ as $n \to \infty$ [9, 10]. In [11], another family of 4-braids, given by $(\sigma_1\sigma_3\sigma_2^{-1})^k$ for $k > 2$, is presented in Example 6.1. However, the author does not compute the meridian length. Instead, they demonstrate that the crossing arcs are isotopic to geodesics by using certain labels and their respective ratios.

In this work, we compute the meridian length for an infinite family of alternating links with two components, specifically those formed by the closure of the braid $\left(\sigma_2^{-1}\sigma_1\sigma_3\sigma_2^{-1}\right)^n$ for odd $n \geq 3$. The diagram of this family is a connected prime alternating diagram and hence by Menasco's [12] result this family of links is hyperbolic. We will give an algebraic expression for the meridian length in terms of the crossing number n.

2 Preliminaries

Definition 2.1. [13, Section 3.1] An *edge* of a link diagram is its segment (an arc) from a crossing to the nearest crossing. Any link diagram with n crossings has $2n$ edges.

Definition 2.2. [14, Definition 4.9] Let M be a 3-manifold with torus boundary. Define a *cusp*, or *cusp neighborhood* of M to be a neighborhood of ∂M homeomorphic to the product of a torus and an interval, $T^2 \times I$. Define a cusp torus to be a torus component of ∂M, or the boundary of a cusp. A hyperbolic structure on M induces an affine structure on the boundary of any cusp of M.

Definition 2.3. [14, Definition 2.18] A *horosphere* about ∞ in $\partial\mathbb{H}^3$ is a plane parallel to $\mathbb{C}$, consisting of points $\{(x + iy, c) \in \mathbb{C} \times \mathbb{R}\}$ where $c > 0$ is constant. Note for any $c > 0$, this plane is perpendicular to all geodesics through ∞. When we apply an isometry that takes ∞ to some $p \in \mathbb{C}$, note a horosphere is taken to a Euclidean sphere tangent to p. By definition, this is a horosphere about p. A horoball is the region interior to a horosphere.

The metric on $\mathbb{H}^3$ induces a metric on a horosphere. For a horosphere $\{(x + iy, c) \in \mathbb{C} \times \mathbb{R}\}$ about ∞, the metric is just the Euclidean metric, rescaled by $\frac{1}{c}$. We may apply an isometry to any horosphere, taking it to one about ∞. Thus the induced metric on any horosphere will always be Euclidean.

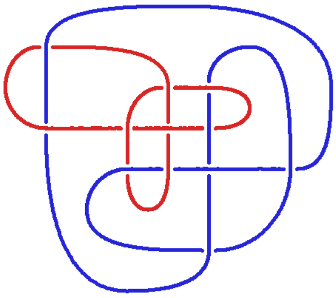
Theorem 2.4. *[14, Proposition 14.1] A complete hyperbolic 3-manifold contains an embedded horoball neighborhood. That is, there is an embedded neighborhood N of the cusps of M such that N lifts to a disjoint collection of embedded horoballs in $\mathbb{H}^3$.*

Lemma 2.5. *[14, Lemma 14.2] Suppose N is an embedded horoball neighborhood of a cusp of M that lifts to the horoball about $\infty \in \partial_\infty\mathbb{H}^3$. Then all the lifts of N to $\mathbb{H}^3$ give countably many horoballs in $\mathbb{H}^3$, with centers at the points $\{g(\infty) \mid g \in \Gamma\}$.*

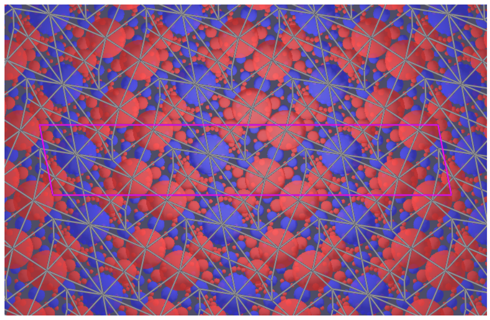
Corollary 2.6. *[14, Corollary 14.3] Let M be finite volume. Any embedded horoball neighborhood about all cusps of M lifts to countably many disjoint horoballs in $\mathbb{H}^3$.*

Definition 2.7. [14, Definition 14.4] A *maximal cusp neighborhood* is an (open) embedded cusp neighborhood for M that is maximal in the sense that no cusp can be expanded while keeping the set of cusps embedded and disjoint.

Definition 2.8. [14, Definition 14.5] Consider the lift of an embedded maximal cusp neighborhood to $\mathbb{H}^3$, with one cusp lifting to a horoball at infinity. A *full-sized horoball* is a horoball in this pattern that is tangent to the horoball at infinity. Viewed from infinity, it has maximal Euclidean diameter.



(a)



(b)

Figure 1: (a): A diagram illustrating the link of two components, created using SnapPy. (b): Horoballs packing of the link complement, also using SnapPy.

Lemma 2.9. *[14, Lemma 14.7] Let M be a hyperbolic 3-manifold with at least one cusp. In a horoball pattern in $\mathbb{H}^3$ given by lifting an embedded maximal cusp neighborhood for M, apply any isometry taking a desired horoball to the one at infinity. Then in the new pattern obtained by applying this isometry, there is at least one full-sized horoball meeting a fundamental domain for the boundary of the horoball about infinity. Moreover, if M has only one cusp, then there are at least two full-sized horoballs in a fundamental domain. The second is often called the Adams horoball.*

3 Main Result : Formula for the Exact Length of the Meridian

The hyperbolic link complement $S^3 - L$ can be lifted in the hyperbolic 3-space $\mathbb{H}^3$. Each component of the link L has a cusp neighborhood (see Definition 2.2) that is homeomorphic to $T \times [1, \infty)$, where T denotes the boundary torus. The preimage of a cusp neighborhood in the link complement within $\mathbb{H}^3$ consists of a set of horoballs. Expanding these horoballs until they are tangent to the horoball at infinity, we obtain what are called full-sized horoballs (see Definition 2.8). When the maximal cusp (see Definition 2.7) of the link complement has finite volume, all cusp neighborhoods lift to countably many disjoint horoballs in hyperbolic 3-space $\mathbb{H}^3$ [14, Corollary 14.3].

Let Γ denote the group of isometries of $\mathbb{H}^3$, and let Γ_∞ be the subgroup consisting of parabolic isometries that fix the point $\{\infty\}$. The fundamental domain of Γ_∞ is represented by a parallelogram in the xy-plane. By selecting the vertices of this parallelogram to coincide with the centers of four full-sized horoballs and aligning one edge with the shortest translation of a parabolic isometry in Γ_∞, the Euclidean length of this edge in the scaled model corresponds to the meridian length l of the torus boundary. The following two results from [15] provide formulas for the diameter of a horoball, which are essential in determining the meridian length l.

Lemma 3.1. *[15, Lemma 2.6] Up to the action of Γ_∞, every horoball other than H_∞ has a horoball of the same diameter paired to it, which is called an associated horoball.*

Lemma 3.2. *[15, Lemma 2.8] Given a horoball H of diameter k in the horoball packing, both it and its associated horoball H' have a pair of horoballs on either side of them with centers at a distance k/l from the centers of H and H', and with diameter k/l^2.*

Expanding the horoball and scaling the horizontal horosphere at height $z = 1$ yields a horoball with the diameter of $1/l^2$ by Lemma 3.2.

Thistlethwaite and Tsvietkova [9, 10], developed an alternative method for computing the complete hyperbolic structure. This construction is based on a link diagram that is taut. Every alternating link admit taut diagram [9, 10], hence the under discussion 4-braid admits a taut diagram. A link diagram consists of crossings and edges. Thistlethwaite and Tsvietkova define the *crossing parameter* w and *edge parameter* u. The crossing parameter w is associated with an arc $\widetilde{\gamma}$ in hyperbolic 3-space $\mathbb{H}^3$. Specifically, the arc $\widetilde{\gamma}$ is a lift of an arc γ situated within the link complement between a crossing. If the arc γ connects the same torus boundary component, then the absolute value of w associated with γ, i.e., $\mid w \mid$ is the Euclidean diameter of the maximal horoball in $\mathbb{H}^3$.

By using hyperbolic geometry, one can analytically derive the Euclidean diameter of the associated horoball. We fix $\mid w \mid = e^{-d}$ where d denotes the hyperbolic distance between two horospheres:

$$\mid w \mid = e^{-d} = e^{-ln\frac{l^2}{1}} = e^{ln\frac{1}{l^2}} = \frac{1}{l^2}$$

Combining the above result with the argument in Lemma 3.2, we conclude that the meridian length l is given by $\mid w \mid^{-\frac{1}{2}}$.

If the arc γ connects distinct torus boundary components, then $\mid w \mid$ represents the diameter of the horoball, with both horoballs expanding simultaneously. These horoballs correspond to the two distinct boundary components joined by γ. Thistlethwaite conjectures that when the horoballs are expanded simultaneously, the meridian length on the boundary torus for alternating links is bounded above by 2. It is worth emphasizing that the conjecture is for when all the horoballs are simultaneously expanded. There are many examples of links where expanding just one horoball results in a meridian length exceeding 2.

Theorem 3.3. *Let B_n be the infinite family of alternating links with two components that is the closure of the braid $\left(\sigma_2^{-1}\sigma_1\sigma_3\sigma_2^{-1}\right)^n$ for odd $n \geq 3$.*

1. *Meridian length of each braid can be computed as $l = |w_1|^{-\frac{1}{2}}$, where w_1 is a root of the following polynomial equation:*

$$\begin{aligned}&\left(16L^4 - 8L^2 + 1\right) w_1^4 + \left(16L^4 + 3L^2 - 2\right) w_1^3\\&+ \left(12L^4 - 2L^2 + 1\right) w_1^2 + \left(4L^4 - 2L^2\right) w_1 + L^4 = 0,\end{aligned}$$

   with $L = \frac{1}{2}\sec\left(\frac{\pi}{n}\right)$.

2. *As n approaches infinity, the meridian length l of the infinite family B_n converges to an algebraic number given by:*

$$\frac{\sqrt{6y}\cdot\sqrt[6]{x}}{\sqrt[4]{-880\cdot y\sqrt[3]{x} - 330\sqrt{33} - 726 + 12\cdot 11^{\frac{2}{3}}x^{\frac{2}{3}} + 11yx^{\frac{4}{3}}}}$$

   where

$$x = 3\sqrt{33} + 77, \quad y = \sqrt[3]{11} \quad ,$$

   which is approximately equal to 2.09355577138662*.*

Proof. A segment of link diagram of 4-braids B_n is drawn in Figure 2. When n is odd, the closure of this 4-braids is link with two components , and when n is even, the closure of this 4-braids is link with four components. For $n = 5$, the closure of the braid B_5 is shown in the Figure 3

Using the algorithm provided by Thistlethwaite and Tsvietkova [9, 10], one can get the following polynomial equations of the infinite family B_n.

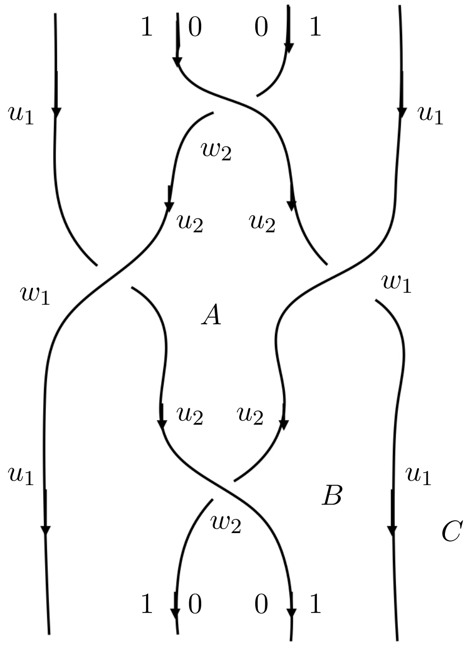


Figure 2: A segment of link diagram of 4-braids B_n .

- The region A has four sides, and we get one equation from it.

$$u_2^2 - w_1 + w_2 = 0$$

- The region B is a four sides, and we get the following equations from it.

$$u_1u_2 + u_1 + u_2 + 1 + w_1 - w_2u_1 - w_2 = 0$$

$$u_2 + 1 - 2w_2 = 0$$

$$u_1u_2 + u_1 + u_2 + 1 + 2w_1 = 0$$

- The region C has n sides.

$$w_1 - L^2u_1^2 = 0$$

where $L = \frac{1}{2}\sec\left(\frac{\pi}{n}\right)$ and L^2 is a shape parameter for regular n-sided polygon. The derivation of this shape parameter can be found in [9, Proposition 2.3]. By using the Gröbner basis method, one can reduce the above system of equations to a single variable polynomial equation:

$$\left(16L^4 - 8L^2 + 1\right) w_1^4 + \left(16L^4 + 3L^2 - 2\right) w_1^3 + \left(12L^4 - 2L^2 + 1\right) w_1^2 + \left(4L^4 - 2L^2\right) w_1 + L^4 = 0$$

This degree four polynomial equation can be solved by using Ferrari's method. There are two Galois conjugate solutions of this polynomial equation. The expressions of the roots of this equation are gigantic, and only compact forms are listed in Appendix A. After finding the solution, we compute the meridian length $l =\mid w_1 \mid^{-\frac{1}{2}}$ on the torus boundary. For odd values of n, w_1 correspond to crossing arc running between the same boundary tori. As $n \longrightarrow \infty$, $L \longrightarrow \frac{1}{2}$ and the meridian length l converges to

$$\frac{\sqrt{6y}\cdot\sqrt[6]{x}}{\sqrt[4]{-880\cdot y\sqrt[3]{x} - 330\sqrt{33} - 726 + 12\cdot 11^{\frac{2}{3}}x^{\frac{2}{3}} + 11yx^{\frac{4}{3}}}}$$

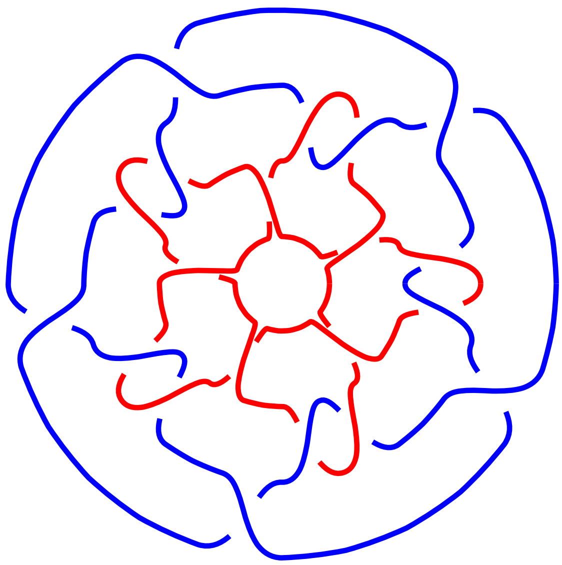


Figure 3: An example of B_5.

where

$$x = 3\sqrt{33} + 77, \quad y = \sqrt[3]{11}$$

which is approximately equal to 2.09355577138662.

□

4 Future Problem

We are interested in finding other geometric invariants of the infinite family of 4-braids B_n. For instance an explicit expression for exact volume like the one for 2-bridge link has been computed in [16]. Given that the canonical decomposition of 2-bridge link is well-known by the work of Sakuma and Weeks [17]. Our preliminary task for the infinite family B_n will be determining the canonical decomposition of the link complement, followed by the derivation of an exact volume expression for this family.

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A Appendix : Algebraic Expression for Meridian Length of Infinite Family B_n

Here we outline Ferrari's method for solving a 4-degree polynomial equation from Section 3.

$$\left(16L^4 - 8L^2 + 1\right) w_1^4 + \left(16L^4 + 3L^2 - 2\right) w_1^3 + \left(12L^4 - 2L^2 + 1\right) w_1^2 + \left(4L^4 - 2L^2\right) w_1 + L^4 = 0$$

Let

- $A = (16L^4 - 8L^2 + 1)$
- $B = (16L^4 + 3L^2 - 2)$
- $C = (12L^4 - 2L^2 + 1)$
- $D = (4L^4 - 2L^2)$
- $E = L^4$

and

- $a = \frac{C}{A} - \frac{3B^2}{8A^2}$
- $b = \frac{D}{A} - \frac{BC}{2A^2} + \frac{B^3}{8A^3}$
- $c = \frac{E}{A} - \frac{BD}{4A^2} + \frac{B^2C}{16A^3} - \frac{3B^4}{256A^4}$
- $P = -\frac{a^2}{12} - c$
- $Q = -\frac{a^3}{108} + \frac{ac}{3} - \frac{b^2}{8}$
- $R = -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}$, either sign of the square root will work.
- $U = \sqrt[3]{R}$, there are 3 complex roots, any one of them will work.
- If $U = 0$ then $y = -\frac{5}{6}a - \sqrt[3]{Q}$
- If $U \neq 0$ then $y = -\frac{5}{6}a + U - \frac{P}{3U}$
- $W = \sqrt{a + 2y}$

and the four roots will be

- $w_1 = -\frac{B}{4A} + \frac{W + \sqrt{-\left(3a + 2y + \frac{2b}{W}\right)}}{2}$
- $w_1 = -\frac{B}{4A} + \frac{W - \sqrt{-\left(3a + 2y + \frac{2b}{W}\right)}}{2}$
- $w_1 = -\frac{B}{4A} + \frac{-W + \sqrt{-\left(3a + 2y - \frac{2b}{W}\right)}}{2}$
- $w_1 = -\frac{B}{4A} + \frac{-W - \sqrt{-\left(3a + 2y - \frac{2b}{W}\right)}}{2}$

Let us define the following intermediate variables:

$$
\begin{aligned}
A &= -3456L^{12} + 3024L^{10} + 2691L^8 - 1708L^6 + 438L^4 - 48L^2 + 2, \\
B &= -42052608L^{20} + 58237920L^{18} - 29571831L^{16} + 6745464L^{14} \\
&\quad - 690444L^{12} + 27216L^{10}, \\
C &= \frac{A + \sqrt{B}}{3 \cdot 2^{1/3}(4L^2 - 1)^2}, \\
D &= \frac{(16L^4 + 3L^2 - 2)^2}{4(4L^2 - 1)^4}, \\
E &= \frac{12L^4 - 2L^2 + 1}{(4L^2 - 1)^2}, \\
F &= \frac{12L^4 - 2L^2 + 1}{3(16L^4 - 8L^2 + 1)}, \\
G &= \frac{2^{1/3}(144L^8 - 84L^6 + 82L^4 - 16L^2 + 1)}{3(4L^2 - 1)^2 C^{1/3}}, \\
H &= \frac{(16L^4 + 3L^2 - 2)^3}{(4L^2 - 1)^6}, \\
I &= \frac{4(12L^4 - 2L^2 + 1)(16L^4 + 3L^2 - 2)}{(4L^2 - 1)^4}, \\
J &= \frac{16(2L^4 - L^2)}{(4L^2 - 1)^2}.
\end{aligned}
$$

Here is the expression for all roots of w_1 in terms of the intermediate variables:

$$
\begin{aligned}
w_1^{(1)} &= -\frac{16L^4 + 3L^2 - 2}{4(4L^2 - 1)^2} - \frac{1}{2}\sqrt{C^{1/3} + D - E + F + G} \\
&\quad - \frac{1}{2}\sqrt{-C^{1/3} + \frac{D}{2} - E - F - G - \frac{H - I + J}{4\sqrt{C^{1/3} + D - E + F + G}}}, \\
w_1^{(2)} &= -\frac{16L^4 + 3L^2 - 2}{4(4L^2 - 1)^2} + \frac{1}{2}\sqrt{C^{1/3} + D - E + F + G} \\
&\quad - \frac{1}{2}\sqrt{-C^{1/3} + \frac{D}{2} - E - F - G + \frac{H - I + J}{4\sqrt{C^{1/3} + D - E + F + G}}}, \\
w_1^{(3)} &= -\frac{16L^4 + 3L^2 - 2}{4(4L^2 - 1)^2} - \frac{1}{2}\sqrt{C^{1/3} + D - E + F + G} \\
&\quad + \frac{1}{2}\sqrt{-C^{1/3} + \frac{D}{2} - E - F - G + \frac{H - I + J}{4\sqrt{C^{1/3} + D - E + F + G}}}, \\
w_1^{(4)} &= -\frac{16L^4 + 3L^2 - 2}{4(4L^2 - 1)^2} + \frac{1}{2}\sqrt{C^{1/3} + D - E + F + G} \\
&\quad + \frac{1}{2}\sqrt{-C^{1/3} + \frac{D}{2} - E - F - G - \frac{H - I + J}{4\sqrt{C^{1/3} + D - E + F + G}}}.
\end{aligned}
$$

The full form of the solution is available on the Mathematica link: https://www.wolframcloud.com/obj/7c9ea2c2-2955-493c-bf76-a7b08cebb391

Here is the expression for meridian length $l = \mid w \mid^{-\frac{1}{2}}$ with the intermediate variables:

$$M = \frac{6 - 9L^2 - 48L^4}{(1-4L^2)^2},$$
$$N = \sqrt{-L^{10}(-6+13L^2)^2(-28+589L^2-4256L^4+9216L^6)},$$
$$O = 1 - 24L^2 + 219L^4 - 854L^6 + \frac{2691}{2}L^8 + 1512L^{10} - 1728L^{12} + \frac{3\sqrt{3}}{2}N,$$
$$P = 2 - 48L^2 + 438L^4 - 1708L^6 + 2691L^8 + 3024L^{10} - 3456L^{12} + 3\sqrt{3}N,$$
$$Q = \sqrt{\begin{aligned} &\frac{-12(1-4L^2)^2(1-2L^2+12L^4)}{(1-4L^2)^4} \\ &+ \frac{3(-2+3L^2+16L^4)^2 + (1-4L^2)^2(4-8L^2+48L^4)}{(1-4L^2)^4} \\ &+ \frac{4(1-4L^2)^2(1-16L^2+82L^4-84L^6+144L^8)}{(1-4L^2)^4 O^{1/3}} \\ &+ \frac{2\cdot 2^{2/3}(1-4L^2)^2 P^{1/3}}{(1-4L^2)^4} \end{aligned}},$$
$$R = \sqrt{\begin{aligned} &\frac{-8(1-4L^2)^4(1-2L^2+12L^4) - 3(2-11L^2-4L^4+64L^6)^2}{(1-4L^2)^6} \\ &+ \frac{2(1-4L^2)^4(1-16L^2+82L^4-84L^6+144L^8)}{(1-4L^2)^6 O^{1/3}} \\ &+ \frac{2^{2/3}(1-4L^2)^4 P^{1/3}}{(1-4L^2)^6} \\ &+ \frac{3\sqrt{3}L^2(72-834L^2+3509L^4-6320L^6+4096L^8)}{(1-4L^2)^6 Q} \end{aligned}}.$$

The final expression becomes:

$$\boxed{l = \frac{2\sqrt{3}}{\sqrt{M - \sqrt{3}Q - \sqrt{6}R}}}$$

https://www.wolframcloud.com/obj/50d78fdd-19f1-4c16-803b-d28efdf43b47

B Appendix : Numerical Approximation for Meridian Length of Infinite Family B_n

We have also computed the numerical approximation for the meridian length of the infinite family B_n. Referring to the equations outlined in Section 3, we provide the following Python code to obtain the numerical approximation for the meridian length of the infinite family B_n.

```
#4-braids for n=3, 5, 7 , .. . 99.

import numpy as np
import math

# The variables are represented as elements of the string
# x[0] = w_1
# x[1] = w_2
# x[2] = u_1
# x[3] = u_2

# Define Sec(x) function
def sec(x):
    return 1/math.cos(x)

# Define the function f
def f(x,n):
    return np.array([x[3]**2 -x[0] +x[1],
                     x[2]*x[3] + x[2] + x[3] +1 +x[0] - x[1]*x[2] -x[1],
                     x[2]*x[3] +x[2] + x[3] +1 +2*x[0],
                     x[3] + 1 -2*x[1],
                     x[0] - 0.25 * (sec(math.pi / n))**2 * x[2]**2])

# Define the Jacobian of f
def J(x,n):
    return np.array([[-1, 1, 0, 2*x[3]],
                     [1, -x[2] - 1, x[3] - x[1] + 1, x[2] + 1],
                     [2, 0, x[3] + 1, x[2] + 1],
                     [0, -2, 0, 1],
                     [1, 0, -0.25 * (sec(math.pi / n))**2 * 2*x[2], 0]])

# Initial guess for x
x0 = np.array([0.5 + 0.5*1j]*4, dtype=np.complex128)

# Iterate until convergence
for n in range(3,99,2):  # iterate over n from 3 to 99 with steps 2.
    while True:
        # Compute the QR decomposition of J(x)
        Q, R = np.linalg.qr(J(x0,n))

        # Compute the solution to the linear system J(x) * dx = -f(x)
        dx = np.linalg.solve(R, np.dot(Q.conj().T, -f(x0,n)))

        # Update
        x1 = x0 + dx
```

n	**SnapPy Values**	**Our Code Values**
3	1.76095258	1.760953
5	1.96289334	1.962893
7	2.02454571	2.024546
9	2.0511218	2.051122
11	2.06489810	2.064898
13	2.0729294	2.072929
15	2.07801089	2.078011
17	2.08142571	2.081426
19	2.083829311	2.083829
21	2.08558431	2.085584
23	2.086904440	2.086904
25	2.08792219	2.087922
27	2.08872326	2.088723
29	2.08936499	2.089365

Table 1: Comparison of the meridian length by SnapPy and our code.

```
    # Check for convergence
    if np.linalg.norm(x1 - x0) < 1e-6:
        break

    x0 = x1

    print(np.round(x1,6))
```

The code is also available in the git repository [18]. After that, we computed the meridian length $|\ w_1\ |^{-\frac{1}{2}}$ and compared these values against those provided by SnapPy, as detailed in Table 1.